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THERMAL CONVECTION IN A CAVITY FILLED WITH A POROUS MEDIUM: A CLASSIFICATION OF LIMITING BEHAVIOURS

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Abstract · Thermally driven flows in a 2-dim. rectangular cavity filled with a fluid-saturated porous medium are considered in the large Rayleigh number limit when the applied temperature gradient is perpendicular to the gravity vector. The standard boundary layer description of these flows is shown to be valid only for $R \gg 10^4 L^2$, where R is the Rayleigh number based on the cavity height and L is the cavity aspect ratio (length/depth). For flows in which $R = O(L^2)$, $L \to \infty$, the horizontal layers merge and a different solution regime exists. A discussion of the relationship between the merged layer limit (R/L^2 fixed, $L \to \infty$) and the classical Hadley cell limit (R fixed, $L \to \infty$) is given. It is found that the intermediate regime R/L fixed, $L \to \infty$, provides the necessary bridge between the merged layer and the Hadley solutions. Bounds on the extent of the various regimes in the (R, L) parameter space are deduced.

The scaling suggested by the merged layer regime gives rise to an alternative correlation of heat transfer data. Existing theories for the Nusselt number are reviewed and shown to be consistent with the new scaling law.

NOMENCLATURE

- a, b, constants, equations (2.7) and 2.8);
- c, constant in Nusselt number expansion, equation (4.5);
- $C(L, R)$, defined in equation (3.4);
- F_0, defines outer horizontal layer solution (Section 2);
- g, acceleration due to gravity;
- h, cavity height;
- k, permeability;
- K, constant, equation (2.16);
- L, aspect ratio (length/height);
- Nu, Nusselt number, equation (4.1);
- $N(R_2)$, scaled Nusselt number, equation (4.4);
- Q, dimensionless energy flux, equation (5.3);
- R, Darcy Rayleigh number, $k\alpha g T'_w h/\kappa\nu$;
- R_1, R/L, equation (3.6);
- R_2, R/L^2, equation (3.1);
- $R_0(\zeta)$, similarity form for the outer horizontal layer solution;
- $\bar{T}$, dimensionless temperature;
- $(\bar{x}, \bar{z})$, dimensionless Cartesian coordinates;
- (x, z), vertical boundary layer coordinates, equation (2.1);
- y, horizontal boundary layer coordinate, equation (2.14).

Greek symbols

- α, coefficient of thermal expansion;
- δ, boundary layer thickness;
- ζ, outer horizontal layer similarity variable, equation (2.18);
- θ, scaled temperature in outer layer, equation (2.13);
- κ, thermal diffusivity;
- λ, characteristic end layer thickness, equation (3.9);
- ν, kinematic viscosity;
- ϕ, scaled stream function in outer layer, equation (2.13);
- $\bar{\psi}$, dimensionless stream function.

Subscripts

- c, core variables;
- ∞, asymptotic limit ($x \to \infty$) for vertical layers;
- h, horizontal layers;
- m, merged layer variables;
- v, vertical layers;
- conv, convective heat transfer contribution;
- diff, diffusive heat transfer contribution.

Superscripts

- $'$, dimensional variables.

1. INTRODUCTION

THIS paper describes some extensions of a recent analysis [1, 2] of natural convection in a 2-dim. rectangular cavity filled with a fluid-saturated porous medium. Some of the analysis is also relevant to flows of electrically conducting fluids [3]. It is assumed that the vertical end walls of the cavity are at fixed but different temperatures, that the horizontal walls are insulated, and that the flow is steady. A first-order description of the entire flow field, including the corner interaction regions, has been given by Blythe, Daniels and Simpkins [1] and Daniels, Blythe and Simpkins [2]. These papers are concerned with the large Rayleigh number limit for a fixed aspect ratio L (length/height). The present paper discusses the consequences of some new scaling laws that were found previously [1, 2], and puts into context other theories that have appeared in the literature.

Subject to the Boussinesq approximation, the governing non-dimensional equations are [1]

$$\nabla^2\bar{\psi} = -R\frac{\partial \bar{T}}{\partial \bar{x}}, \tag{1.1}$$

$$\frac{\partial(\bar{T},\bar{\psi})}{\partial(\bar{x},\bar{z})} = \nabla^2\bar{T}. \tag{1.2}$$

Here R is the Rayleigh number based on the cavity height h, and the origin of the coordinates, made dimensionless with respect to h, is at the bottom of the cold wall. In equations (1.1) and (1.2), $\bar{\psi}$ is the stream function and $\bar{T}$ is the temperature. Appropriate boundary conditions are

$$\begin{aligned} \bar{\psi} = 0, \bar{T} = 0 \quad &\text{on} \quad \bar{x} = 0, \\ \bar{\psi} = 0, \bar{T} = 1 \quad &\text{on} \quad \bar{x} = L \end{aligned} \tag{1.3}$$

and

$$\bar{\psi} = \frac{\partial \bar{T}}{\partial \bar{z}} = 0 \quad \text{on} \quad \bar{z} = 0, 1. \tag{1.4}$$

Note that the above equations and boundary conditions satisfy the symmetry relations

$$\begin{aligned} \bar{\psi}(\bar{x},\bar{z}) &= \bar{\psi}(L-\bar{x}, 1-\bar{z}), \\ \bar{T}(\bar{x},\bar{z}) &= 1-\bar{T}(L-x, 1-z). \end{aligned} \tag{1.5}$$

2. THE VERTICAL AND HORIZONTAL BOUNDARY LAYERS

Standard scalings [3, 4] for the vertical boundary layer structure on the cold wall at $\bar{x} = 0$ are

$$\bar{\psi}(\bar{x},\bar{z};R) = R^{1/2}\psi(x,z)+\cdots, \quad \bar{T}(\bar{x},\bar{z};R) = T(x,z)+\cdots$$

with (2.1)

$$x = R^{1/2}\bar{x}, \bar{z} = z.$$

This transformation leads to the vertical boundary equations given in ref. [1]. A corresponding boundary layer analysis for the hot wall at $\bar{x} = L$ is avoided by using the symmetry relations (1.5). At the outer edge of the layer the vertical velocity component vanishes and

$$T \to T(\infty, z) = T_\infty(z), \quad \psi \to \psi(\infty, z) = \psi_\infty(z), \quad \text{as} \quad x \to \infty. \tag{2.2}$$

In the core, where $\bar{x}$ and $\bar{z}$ are $O(1)$,

$$\bar{\psi} = R^{1/2}\psi_c(\bar{x},\bar{z})+\cdots, \quad \bar{T} = T_c(\bar{x},\bar{z})+\cdots. \tag{2.3}$$

It follows from equations (1.1) and (1.2) that in the core

$$\frac{\partial T_c}{\partial \bar{x}} = \frac{\partial \psi_c}{\partial \bar{x}} = 0, \tag{2.4}$$

so that matching with the outer limit of the vertical boundary layer solution gives

$$\psi_c(\bar{x},\bar{z}) = \psi_\infty(\bar{z}) = \psi_\infty(z),$$

$$T_c(\bar{x},\bar{z}) = T_\infty(\bar{z}) = T_\infty(z), \tag{2.5}$$

since $\bar{z} = z$.

The evaluation of $\psi_\infty(z)$ and $T_\infty(z)$ requires a detailed analysis of the vertical boundary layer equations together with an additional constraint on ψ_∞. This constraint should come from a discussion of the core behavior near the horizontal boundaries. It is usually assumed [3–5] that there is negligible mass flux in the horizontal layers adjacent to the boundaries or, equivalently, that the vertical boundary layers empty into the core, i.e.

$$\psi_\infty(0) = \psi_\infty(1) = 0. \tag{2.6}$$

An alternative assumption, that the vertical energy flux across the core is zero, has been made by Bejan [6]. However, this requirement reduces to equation (2.6) in the limit $R \to \infty$ at fixed L [7]. The analysis in refs. [1, 2] also demonstrates that in this limit (2.6) is consistent with the horizontal and vertical boundary layer structures. Further discussion on the Bejan hypothesis [6] can be found in Section 5.

A consequence of the analysis given in ref. [1] is that

$$\psi_\infty(z) \sim az^{1/2} \quad \text{and} \quad T_\infty(z) \sim bz^{1/2} \quad \text{as} \quad z \to 0 \tag{2.7}$$

where

$$a \simeq 1.616 \quad \text{and} \quad b \simeq 0.270. \tag{2.8}$$

Note that the parameters a and b are independent of L. Similarly, from the symmetry conditions it follows that

$$\psi_\infty(z) \sim a(1-z)^{1/2}$$

and (2.9)

$$T_\infty(z) \sim 1-b(1-z)^{1/2} \quad \text{as} \quad z \to 1.$$

Equations (2.7) and (2.9) are clearly consistent with equation (2.6).

A measure of the vertical boundary layer thickness δ_v can be obtained from an earlier analysis [7] which suggests, for a suitable two-layer model, that

$$\delta_v(z) = 8\psi_\infty/(9R^{1/2}T_\infty). \tag{2.10}$$

Consequently, on the center-line $z = 1/2$,

$$\delta_v(1/2) = 1.305R^{-1/2}. \tag{2.11}$$

If the total vertical boundary layer thickness is to be small compared with the cavity length [4], i.e. $2\delta_v < L$,

$$R > 6.81L^{-2}. \tag{2.12}$$

An equivalent criterion for a Newtonian fluid was first given by Gill [5].

Equations (2.7) and (2.9) have important ramifications concerning the structure of the solutions near the horizontal surfaces. The detailed analysis given in ref. [2] demonstrates that, away from the corners, the horizontal structure splits into two regions. In the outer region, where thermal diffusion is not important, appropriate scalings are

$$\bar{\psi} = R^{3/8}\phi(\bar{x}, y) + \cdots, \quad \bar{T} = R^{-1/8}\theta(\bar{x}, y) + \cdots \tag{2.13}$$

and

$$\bar{z} = R^{-1/4}y. \tag{2.14}$$

It can be shown that ([2], Section 3)

$$\theta = K\phi = L^{1/4}F_0(\bar{x}/L, y/L^{1/2}) \tag{2.15}$$

where

$$K = b/a \simeq 0.167. \tag{2.16}$$

The function F_0 is defined by

$$F_0 = \left(\frac{\pi^2}{2K}\right)^{1/4} b(1-\bar{x}/L)^{1/4}R_0(\zeta) \tag{2.17}$$

where

$$\zeta = yL^{-1/2}[2K^{-1}(1-\bar{x}/L)]^{-1/2} \tag{2.18}$$

and R_0 can be expressed in terms of parabolic cylinder functions. As $\zeta \to \infty$ the behavior of $R_0(\zeta)$ is algebraic. The exponential decay length associated with the approach to this algebraic behavior corresponds to ([2], equation (3.22))

$$\zeta = 2, \tag{2.19}$$

From equation (2.14) the equivalent horizontal layer thickness is

$$\delta_h = (y)_{\zeta=2}R^{-1/4}. \tag{2.20}$$

Hence, from equation (2.18),

$$\delta_h = 2[L/KR^{1/2}]^{1/2} \quad \text{at} \quad \bar{x} = L/2. \tag{2.21}$$

If the total thickness of the horizontal layers is to be small compared with the cavity height, i.e. $2\delta_h < 1$, then

$$R > \left(\frac{16}{K}\right)^2 L^2 \simeq 9.17 \times 10^3 L^2. \tag{2.22}$$

This is one of the principal consequences of the analysis in refs. [1, 2]. Clearly, when $L > 1$ the limitation imposed by the above condition (2.22) is very strong. In particular, this result implies that for $L > 1$ the boundary layer description will be valid only if $R \gg 10^4$. When $R \lesssim 10^4$ the horizontal layers merge, and vertical diffusion becomes important in the core region.

The restrictions defined by equation (2.12) for the vertical layers and by equation (2.22) for the horizontal layers are displayed in Fig. 1. Although the vertical boundary layers on the end walls are thin even for moderate values of R, the importance of the horizontal layers cannot be overstressed and, for $L > 1$, it appears that the boundary layer approximation will be of limited use.

3. LARGE ASPECT RATIO LIMITS

The discussion given in Section 2 suggests that an important distinguished limit arises as

$$L \to \infty \quad \text{with} \quad R_2 = \frac{R}{L^2} \text{ fixed.} \tag{3.1}$$

In this merged layer régime, where the horizontal layers apparently fill the cavity, it is expected that diffusive and convective effects will be important across the depth of the cavity but that longitudinal diffusion can be ignored. Suitable expansions are

$$\bar{\psi} = L\psi_m(\bar{x}/L, z; R_2) + \cdots$$

and

$$\bar{T} = T_m(\bar{x}/L, z; R_2) + \cdots \tag{3.2}$$

where the subscript m indicates the merged régime. The scaled equations must be supplemented by appropriate end layers in which longitudinal diffusion is important. These vertical end layer equations are equivalent to the

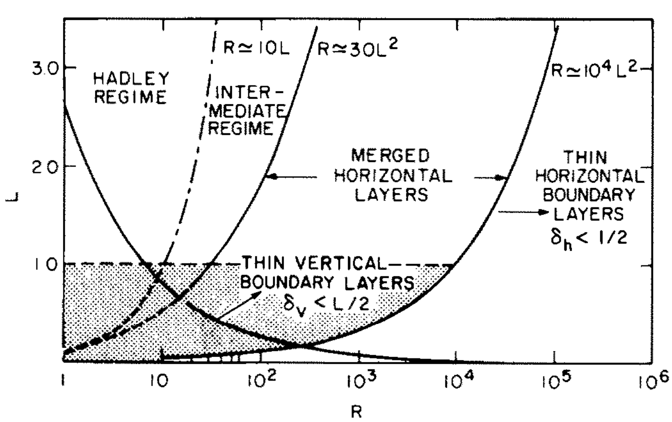

FIG. 1. Regimes of various limiting behaviours.

standard vertical boundary layer equations discussed above. A detailed discussion and analysis of the equations governing ψ_m and T_m will be given in a future paper [8].

For the limit (3.1), the end layers on the vertical walls are of the same form as in the boundary layer limit, $R \to \infty$ at fixed L. It is important to note that the boundary layer structure is embedded in the merged layer structure and can be recovered by the subsequent limit $R_2 \to \infty$. As noted earlier, the result (2.22) suggests that the boundary layer régime requires $R_2 \gg 10^4$; if $R_2 \lesssim 10^4$ and $L \gg 1$ the merged layer structure must be used. Within (R, L) parameter space, the complete extent of the merged layer régime is not obvious. Solutions in the limit $L \to \infty$ at fixed R are known [9]. For a Newtonian fluid the problem corresponds to what is termed a Hadley cell in geophysics, and has been studied [10]. In this limit ($L \to \infty$, R fixed) the stratified core flow is driven by a constant horizontal temperature gradient and is parallel with the horizontal walls: the flow is turned in end layers near the vertical boundaries. The connection between the Hadley régime, as R increases, and the merged layer régime, as R_2 decreases, is discussed below.

At fixed R, $L \to \infty$, the solution for the core stream function has the form [9]

$$\bar{\psi} = \frac{RC}{2L} z(1-z). \tag{3.3}$$

In general $C = C(L; R)$ and a series representation, $L \to \infty$, has been given [8]. The expansion can be rewritten as

$$C = 1 + L^{-1}\left(\hat{C}_3 \frac{R^2}{L^2} + \hat{C}_5 \frac{R^4}{L^4} + \cdots\right) \tag{3.4}$$

where

$$\hat{C}_3 \simeq -9.9 \times 10^{-3}, \quad \hat{C}_5 \simeq 1.0 \times 10^{-4}. \tag{3.5}$$

Obviously, the expansion fails when

$$R_1 = R/L = O(1), \tag{3.6}$$

even though $(C-1)$ is still small, i.e. $(C-1) = O(L^{-1}) = o(1)$. Equation (3.5), and other results in ref. [8], suggest that the upper limit for the Hadley régime is

$$R_1 \sim 10; \tag{3.7}$$

this result is shown in Fig. 1.

The restriction (3.6) is clearly distinct from the limitation associated with the merged layer régime, $R_2 = O(1)$, and suggests the existence of an intermediate régime defined by

$$L \to \infty \text{ with } R_1 \text{ fixed}. \tag{3.8}$$

Equations corresponding to this limit have been derived; their properties are described elsewhere [8]. It is found that in the core the leading approximation still corresponds to the Hadley structure, but the higher-order terms are different from those defined by equations (3.3) and (3.4) for the fixed R limit. Furthermore, for $R_1 = O(1)$ the end layer structure is more complex and is essentially governed by the full Boussinesq equations. Although an analytical solution of the problem is not in general possible, the form of the asymptotic decay in the end layers can be determined. It can be shown that the approach of the end layer solution towards the core solution introduces a characteristic length $\lambda(R_1)$ for the end layer thickness. The determination of λ requires the solution of a fourth-order eigenvalue problem [8], but approximate analysis gives*

$$\lambda \simeq R_1/60, \quad R_1 \gg 1. \tag{3.9}$$

Consequently, the end layers spread into the interior of the cavity unless

$$\frac{R_1}{60} \lesssim \frac{L}{2}$$

i.e.

$$\frac{R_1}{L} = R_2 \lesssim 30, \tag{3.10}$$

which defines the lower limit of the merged layer régime (Fig. 1). As R_1 increases, the structure of the end layers in the intermediate régime splits into two zones [8] with an inner wall zone corresponding to the standard vertical boundary layer structure [1]. The outer zone is associated with the core structure for the merged régime in which both convection and vertical diffusion are important.

4. PARAMETER DEPENDENCE OF THE NUSSELT NUMBER

A dimensionless measure of the heat flux across the end wall $\bar{x} = 0$ is defined by the Nusselt number

$$Nu = L\int_0^1 \left(\frac{\partial T}{\partial x}\right)_{x=0} dz. \tag{4.1}$$

Since the horizontal boundaries are adiabatic the heat flux across any vertical cross section must be the same. It follows from the energy equation (1.2) that equation (4.1) is equivalent to [9]

$$Nu = L\int_0^1 \left\{\frac{\partial \bar{T}}{\partial x} - \frac{\partial \bar{\psi}}{\partial z}\bar{T}\right\} dz. \tag{4.2}$$

A number of efforts to correlate the dependence of Nu on the parameters R and L have been made. Recent work has been summarized by Bejan [11]. For $L > 1$ the discussion given in Section 3 suggests that much of the data will be in the merged layer régime ($L \to \infty$ at fixed R_2).

For this régime the expansions (3.2) imply that, to a first approximation, (4.2) reduces to

$$Nu = L^2 \int_0^1 \left(-\frac{\partial \psi_m}{\partial z}\right) T_m \, dz \tag{4.3}$$

* A more accurate estimate of the smallest positive eigenvalue is 64.9.

which must be independent of $\bar{x}/L$. Consequently,

$$Nu = L^2 N(R_2) \quad (4.4)$$

i.e. Nu/L^2 should depend only on R_2 in the merged layer régime.

Since the boundary layer solution is included in the merged layer structure [8], it follows that the behavior of N as $R_2 \to \infty$ is known. In this limit $\psi_m \to R_2^{1/2}\psi_\infty$ and $T_m \to T_\infty$, where ψ_∞ and T_∞ are independent of R_2. Numerical calculations [9, 12] give

$$N(R_2) \sim cR_2^{1/2} \simeq 0.515\, R_2^{1/2}, \quad \text{as} \quad R_2 \to \infty. \quad (4.5)$$

As R_2 decreases the merged layer analysis fails, and for $R_2 = O(L^{-1})$ the intermediate régime defined by (3.6) provides the correct description. Unlike the boundary layer limit, the intermediate core structure is not contained in the merged layer equations. However, the limiting behavior as $R_2 \to 0$ of $N(R_2)$ can be obtained from an analysis of the intermediate régime in the secondary limit $R_1 \to \infty$. From the intermediate equations [8] it can be shown that

$$Nu = \frac{R_1^2}{120} + 1 + O(L^{-1}). \quad (4.6)$$

The leading terms in (4.6) are also contained in the Hadley cell expansion [9, 13]. As $R_1 \to \infty$ it is seen that equation (4.6) implies

$$N \sim \frac{R_2^2}{120} \quad \text{as} \quad R_2 \to 0. \quad (4.7)$$

The estimate given in equation (3.10) suggests that the merged layer expansion will fail when $R_2 \simeq 30$. It is anticipated that (4.7) will not provide a good numerical approximation at the lower bound of the merged layer régime where higher-order terms in the expansion (4.7) will be required.

5. DISCUSSION OF EARLIER WORK

Although the above estimates suggest that equation (4.4) should be approximate for $R_2 > 30$, the general form of the function $N(R_2)$ is unknown. As $R_2 \to \infty$ boundary layer analysis gives the limiting behaviour defined by equation (4.5) which is shown in Fig. 2. Strictly, the merged layer solution is associated with $L > 1$, but for sufficiently large R it is expected that the boundary layer limit is also appropriate for $L < 1$. In particular, from equation (2.22), the boundary layer description is valid when $R_2 \gg 10^4$. Consequently, the curve defined by equation (4.5) represents the asymptotic form of Nu irrespective of L. At the lower limit of the merged layer régime, the structure of the intermediate régime implies that N must become L-dependent as $R_2 \to 0$.

An earlier analysis by Bejan and Tien [13] considered the limit $L \to \infty$. Their approach was based on the assumption that the flow away from the end walls is parallel with the horizontal boundaries. (In this sense the analysis is similar to the work by Cormack, Leal and Imberger [10] on the Newtonian fluid problem.) Near the vertical boundaries a polynomial profile was used to represent the local behaviour. As $L \to \infty$ it can be shown from the Bejan–Tien analysis [13] that

$$Nu/L^2 \sim N_\alpha(R_2) = (\sqrt{30}/4)R_2^{-1}[(1+2R_2/\sqrt{30})^{1/2}-1]^3 \quad (5.1)$$

where

$$N_\infty \sim 0.302\, R_2^{1/2} \quad \text{as} \quad R_2 \to \infty. \quad (5.2)$$

Equation (5.2) is not a good quantitative representation of the true asymptotic behavior [equation (4.5)], but it is remarkable that equations (5.1) and (5.2) are qualitatively consistent with the merged layer prediction. The function defined by equation (5.1) is shown in Fig. 2. It should be stressed, however, that in the merged layer régime the core flow is not parallel with the horizontal boundaries and thus the solution proposed by Bejan and Tien cannot be correct in that régime.

In both the intermediate and the Hadley régimes the core flow is parallel with the horizontal boundaries.

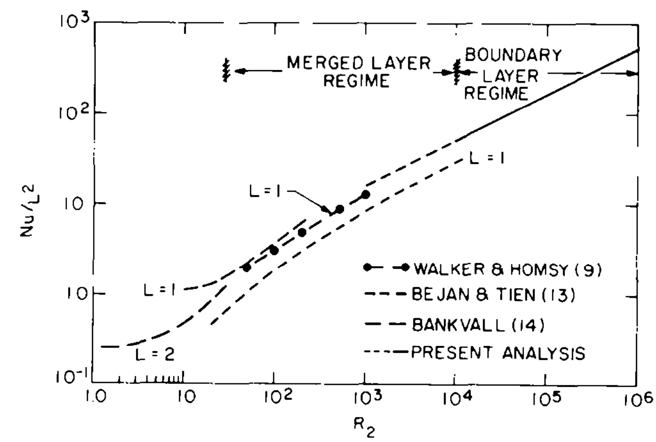

FIG. 2. Correlation for heat transfer data in the merged layer and boundary layer regimes.

However, in these solutions the horizontal temperature gradient in the core is significant, and vertical diffusion is important across the core depth. This behaviour should be contrasted with that in the boundary layer régime where, although the core flow is parallel with the horizontal surfaces, diffusive effects are confined to thin boundary layers and the horizontal temperature gradient across the core is negligible.

The Bejan–Tien theory [13] can be justified in the Hadley limit $L \to \infty$ at fixed R. A more formal treatment of this limit has been given by Walker and Homsy [9], where there is also a discussion of the limit $R \to 0$ at fixed L, i.e. the conduction dominated régime. Walker and Homsy extended the range of validity of the conduction solution to larger values of R by means of Padé approximants and related techniques. In particular, they required that their solution have the asymptotic form (4.5), but calculations were presented only for $L = 1$ and $L = \frac{1}{2}$. The results for $L = 1$ are shown Fig. 2. Within the merged layer régime any L-dependence of the reduced Nusselt number N is associated with the higher-order terms. If this dependence is weak ($L > 1$) the Walker–Homsy curve for $L = 1$ should provide a good approximation to N over the merged layer range. Support for this view follows from numerical calculations by Bankvall [14] which are also in good agreement with the experimental observations of Klarsfeld [15]. Bankvall's results are also shown in Fig. 2 for $L = 1$ and for $L = 2$; the range of R lies between the intermediate and the merged layer régimes. Both curves are consistent with the lower limit of the extended series [9]. Consequently, the Walker–Homsy extension does appear to bridge the gap between the known boundary layer limit ($R_2 \to \infty$) and the numerical calculations which are appropriate to the lower limit of the merged layer régime. Nevertheless, since $L = 1$ for the extended curve shown in Fig. 2, care must be taken in assuming universal validity over the merged layer range.

At lower values of R_2, in the intermediate régime, the L-dependent nature of Bankvall's data is evident (Fig. 2). Since the governing parameters are now R_1 and L this dependence can be anticipated, and is confirmed by the recent calculations of Hickox and Gartling [16]. Their results lie predominantly in the intermediate régime, but the value of N obtained for a case that is in the merged layer range agrees closely with the earlier work [9, 14].

The boundary layer limit (4.5) for the Nusselt number has been discussed at length in the literature. Weber [4] used a modified Oseen approach [5] to deduce that $c \simeq 0.58$, which is somewhat greater than the value predicted by the numerical calculations [9, 12] [equation (4.5)]. Bejan [6] attempted to improve the Weber result by replacing the core condition (2.6) by the requirement that the vertical energy flux vanish. Although this suggestion appears to give better agreement with numerical solutions of the full equations, it has been argued [7] that a more accurate integration of the boundary layer equations also improves the agreement between the boundary layer approximation and the Boussinesq equations. As noted earlier, the boundary layer model is valid only if $R_2 \gg 10^4$. When $R_2 < 10^4$ the thick horizontal layers have a pronounced effect on the core solution. Consequently, it is of interest to ask whether the improvement given by the Bejan approach [6] correctly predicts the magnitude of the error terms associated with the horizontal layers.

In terms of the variables introduced in Section 1 the vertical energy flux

$$\bar{Q}(\bar{z}; R, L) = \int_0^L \left[\left(-\frac{\partial \bar{\psi}}{\partial \bar{x}}\right)\bar{T} - \frac{\partial \bar{T}}{\partial \bar{z}}\right] \mathrm{d}\bar{x}. \quad (5.3)$$

For the boundary layer approximation described in Section 2 ($R \to \infty$ at fixed L) the dominant contribution to $\bar{Q}$ comes from the convective transfer along the vertical end layers. This effect is represented by

$$\bar{Q}_{\mathrm{conv}} = \int_0^L \left(-\frac{\partial \bar{\psi}}{\partial \bar{x}}\right)\bar{T}\, \mathrm{d}\bar{x} = O(R^{1/2}) \quad (5.4)$$

in the boundary layer limit. The remaining term in equation (5.3) is associated with vertical diffusion and the major contribution to this term comes from the core solution. Consequently,

$$\bar{Q}_{\mathrm{diff}} = \int_0^L \left(-\frac{\partial \bar{T}}{\partial \bar{z}}\right) \mathrm{d}\bar{x} = O(L). \quad (5.5)$$

For flows in which both L and R are large, equations (5.4) and (5.5) imply that convective and diffusive effects become comparable when

$$R_2 = R/L^2 = O(1), \quad (5.6)$$

which is the quantity used to describe the merged layer limit. The existence of this group was noted by Bejan [6] in a discussion of the correct horizontal boundary condition for the core solution. Bejan suggested that (2.6) should be replaced by a requirement that the net vertical energy flux, as evaluated from the core and vertical boundary layer solutions, must vanish as $\bar{z} \to 0, 1$. Although for adiabatic walls $\bar{Q} \equiv 0$ on the horizontal boundaries, Bejan's hypothesis requires that $\bar{Q}$ must vanish at the outer edge of the horizontal layers or, equivalently, that the vertical energy flux across the horizontal layers is negligible [c.f. $O(R^{1/2})$].

For the boundary layer limit, $R_2 \gg 1$, Bejan noted that the energy flux hypothesis reduces to equation (2.6). In fact, the hypothesis is correct only in this limit. From the outer horizontal layer solution described in Section 2

$$\bar{Q}_{\mathrm{conv}} = O(R^{1/4}L^{1/2}), \quad \bar{Q}_{\mathrm{diff}} = O(R^{1/8}L^{3/4}). \quad (5.7)$$

Hence, as $R \to \infty$ at fixed L, $\bar{Q} = o(R^{1/2})$ within the horizontal layer and the Bejan hypothesis is valid. Alternatively, if $R_2 = O(1)$, the horizontal layers merge (Section 3), the boundary layer concept is not appropriate, and the energy flux hypothesis fails. In this case the estimates (5.7), together with equation (5.6), indicate that

$$\bar{Q} = O(\bar{Q}_{\mathrm{conv}}) = O(\bar{Q}_{\mathrm{diff}}) = O(R^{1/2}). \tag{5.8}$$

6. SUMMARY

The main conclusions of the analysis are listed below.

(1) A boundary layer model for the flow field is valid only if $R_2 = RL^{-2} \gg 10^4$. When $L > 1$ this provides a strong restriction on the magnitude of the Rayleigh number.

(2) For $L > 1$, the various solution regimes correspond (approximately) to

(i) $30L^2 < R < 10^4 L^2$, merged layer;
(ii) $10L < R < 30L^2$, intermediate;
(iii) $R < 10L$, Hadley.

(3) Within the merged layer regime the heat transfer law can be reduced to

$$Nu = L^2 N(R_2).$$

Existing theories are consistent with this new scaling law, i.e. Nu/L^2 is dependent only on R_2.

(4) The vertical energy flux hypothesis [6] is valid only in the boundary layer limit, $R_2 \to \infty$; it does, however, generate the merged layer similarity group.

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CONVECTION THERMIQUE DANS UNE CAVITE REMPLIE PAR UN CORPS POREUX: UNE CLASSIFICATION DES COMPORTEMENTS LIMITES

Résumé—Des écoulements mus par la chaleur dans une cavité rectangulaire bidimensionnelle, remplie d'un milieu poreux saturé en fluide, sont considérées dans une limite de grands nombres de Rayleigh quand le gradient de température appliqué est perpendiculaire au vecteur gravité. La description classique de couche limite est valide seulement si $R \gg 10^4 L^2$, où R est le nombre de Rayleigh basé sur la hauteur de la cavité et où L est le rapport de forme (longueur/profondeur). Pour des écoulements avec $R = 0\,(L^2)$, $L \to \infty$, les couches horizontales disparaissant et un régime nouveau de solution existe. On donne une discussion de la relation entre la limite de couche (R/L^2 fixé et $L \to \infty$) et la limite de la cellule de Hadley classique (R fixé, $L \to \infty$). On trouve que le régime intermédiaire R/L fixé, $L \to \infty$, fournit le pont nécessaire entre les solutions de couche et de Hadley. On déduit des extensions des différents régimes dans l'espace (R, L).

L'étude du régime donne une corrélation alternative des données de transfert thermique. Des théories existantes pour le nombre de Nusselt sont revues et on montre qu'elles sont cohérentes avec la nouvelle loi de structure.

THERMISCHE KONVEKTION IN EINEM MIT PORÖSEM MEDIUM GEFÜLLTEN HOHLRAUM: EINE KLASSIFIKATION VON GRENZZUSTÄNDEN

Zusammenfassung Thermisch bedingte Strömungen in einem zweidimensionalen, rechtwinkligen, mit flüssigkeitsgesättigtem porösen Medium gefüllten Hohlraum wurden in dem großen Bereich von Rayleigh-Zahlen untersucht, in welchem ein zum Schwerkraftvektor senkrechter Temperaturgradient auftritt. Es wird gezeigt, daß die übliche Grenzschichtbeschreibung dieser Strömungen nur für $R \gg 10^4 L^2$ gültig ist, wobei R die mit der Hohlraumhöhe gebildete Rayleigh-Zahl und L das Seitenverhältnis (Länge/Tiefe) des Hohlraums ist. Für strömungen, bei denen $R = O(L^2)$ und $L \to \infty$ gilt, vermischen sich die horizontalen Schichten, und es existiert ein anderer Lösungsbereich. Die Beziehung zwischen den Grenzbedingungen für die Vermischung der Schichten (R/L^2 konstant, $L \to \infty$) und für das klassische Hadley-Zellenmodell (R konstant, $L \to \infty$) wird diskutiert. Dabei ergibt sich, daß der Zwischenbereich (R/L konstant, $L \to \infty$) die erforderliche Verbindung zwischen dem Bereich der Vermischung der Schichten und der Hadley-Lösung darstellt. Grenzen der verschiedenen Bereiche im (R, L)-Parameterraum werden hergeleitet. Die für den Bereich der vermischten Schichten zweckmäßige Darstellung führt zu einem neuen Ansatz für die Korrelation von Wärmeübertragungsdaten. Vorliegende Theorien für die Bestimmung der Nusselt-Zahl werden darauf hin überprüft und ihre Verträglichkeit mit der neuen Ähnlichkeitsbeziehung festgestellt.

ТЕПЛОВАЯ КОНВЕКЦИЯ В ПОЛОСТИ, ЗАПОЛНЕННОЙ ПОРИСТОЙ СРЕДОЙ. КЛАССИФИКАЦИЯ ПРЕДЕЛЬНЫХ РЕЖИМОВ

Аннотация—Возникающие под действием градиента температур течения в двумерной прямоугольной полости, заполненной пористой средой, которая насыщена жидкостью, исследуются в пределе большого значения числа Рэлея. Градиент температур направлен перпендикулярно вектору силы тяжести. Показано, что обычное описание этих течений с помощью теории пограничного слоя справедливо только при $R \gg 10^4 L^2$, где R — число Рэлея, отнесенное к высоте полости, а L — отношение ее сторон (длины к глубине). В потоках, характеризующихся значениями $R = O(L^2)$, $L \to \infty$, горизонтальные слои сливаются и возникает режим, описание которого требует иного подхода. Обсуждается соотношение между предельным случаем слоя, образованного в результате такого слияния (R/L^2 постоянное, $L \to \infty$) и предельным случаем классической ячейки Хэдли. Найдено, что промежуточный режим, когда отношение R/L постоянно, а $L \to \infty$, является тем необходимым звеном, которое связывает два вышеупомянутых предельных решения. Определены границы протяженности различных режимов в пространстве, описываемом параметрами R и L. Масштабные параметры слоя, образованного в результате слиения, позволяют получить новую обобщенную зависимость для описания данных по теплообмену, которая хорошо согласуется с имеющимися аналитическими зависимостями для числа Нуссельта.